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ON THE SPECTRAL SEQUENCE ASSOCIATED TO A MULTICOMPLEX

MURIEL LIVERNET, SARAH WHITEHOUSE, AND STEPHANIE ZIEGENHAGEN

Abstract. A multicomplex, also known as a twisted chain complex, has an associated spectral sequence via a filtration of its total complex. We give explicit formulas for all the differentials in this spectral sequence.

1. Introduction

A multicomplex is an algebraic structure generalizing the notion of a (graded) chain complex and that of a bicomplex. The structure involves a family of higher "differentials" indexed by the non-negative integers, and is also known as a twisted chain complex, or a D_∞-module. Multicomplexes have arisen in many different places and play an important role in homotopical and homological algebra. These objects were first considered by Wall [Wa61] in his work on resolutions for extensions of groups and they were studied by Gughenheim and May [GM74] in their approach to differential homological algebra.

A multicomplex has an associated total complex, with filtration, and thus an associated spectral sequence. This spectral sequence plays a key role in the homotopy theory of these objects, as studied in [CELW18a]. The spectral sequence was studied by Boardman [Bo99], and by Hurtubise [Hu10], who noted that the differentials of the spectral sequence differ from the maps induced by the higher "differentials" of the multicomplex. The main content of this short note is to give explicit formulas for all the differentials in this spectral sequence. This description generalizes well-known results in the bicomplex case (see for example [CFUG97]).

We give some examples, revisiting those given by Hurtubise and Wall, and we briefly note some applications. In particular, a new application appears in the recent work of Cirici and Wilson [CW18]. They use our description of the E_2 page of the spectral sequence, in the case of a multicomplex with only four non-zero structure maps, to introduce and study a new invariant for almost complex manifolds, which generalizes the definition of Dolbeault cohomology for complex manifolds.

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Conventions. Throughout the paper k will be a commutative unital ground ring.

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2. The spectral sequence associated to a multicomplex

We begin by introducing multicomplexes, including notation and grading conventions.

Definition 2.1. A *multicomplex* (also called a *twisted chain complex*) is a $(\mathbb{Z},\mathbb{Z})$-graded k-module C equipped with maps $d_i\colon C \to C$ for $i \geqslant 0$ of bidegree $|d_i| = (-i, i-1)$ such that

$$\sum_{i+j=n} d_i d_j = 0 \quad \text{for all } n \geqslant 0.$$

A morphism $f\colon (C, d_i) \to (C', d'_i)$ of multicomplexes is given by maps $f_i\colon C \to C'$ for $i \geqslant 0$ of bidegree $|f_i| = (-i, i)$ satisfying

$$\sum_{i+j=n} f_i d_j = \sum_{i+j=n} d'_i f_j \quad \text{for all } n \geqslant 0.$$

For C a multicomplex and $(a, b) \in \mathbb{Z} \times \mathbb{Z}$, we write $C_{a,b}$ for the k-module in bidegree (a, b).

Remark 2.2. Multicomplexes form a category, $\mathrm{tCh_k}$, with objects and morphisms as in Definition 2.1. Sometimes different sign conventions are adopted. A common alternative is to require the structure maps to satisfy the relations

$$\sum_{i+j=n} (-1)^i d_i d_j = 0 \quad \text{for all } n \geqslant 0,$$

with a similar sign change for the morphisms. It may be checked that the resulting category is isomorphic to $\mathrm{tCh_k}$.

Various other grading conventions may be found, too, such as a single $\mathbb{N}$ or $\mathbb{Z}$ grading, or an $(\mathbb{N},\mathbb{Z})$-grading. We will discuss where our choice of $(\mathbb{Z},\mathbb{Z})$-grading is significant below.

Remark 2.3. It is shown in [LV12, 10.3.17] (singly graded version) and in [LRW13] (bigraded version) that multicomplexes can be viewed as D_∞-algebras, where D is the operad of dual numbers. This point of view is also related to the work of Lapin [La01].

Example 2.4. If the structure maps of a multicomplex satisfy $d_i = 0$ for $i \geqslant 1$, we retrieve the notion of a chain complex with an additional grading, sometimes referred to as a vertical bicomplex. If $d_i = 0$ for $i \geqslant 2$, we retrieve the notion of a bicomplex.

A multicomplex gives rise to a chain complex via totalization. Since we consider $(\mathbb{Z},\mathbb{Z})$-gradings it is a priori not clear which is the right notion of total complex in this setting. See [Me78] for a discussion of this. One could for example associate to a multicomplex C the direct sum total complex with $\bigoplus_{a+b=n} C_{a,b}$ in degree n. It will turn out that the associated spectral sequence has better convergence properties for the following version of the total complex.

Definition 2.5. For a multicomplex C, the *associated total complex* $\mathrm{Tot}C$ is the chain complex with

$$(\mathrm{Tot}C)_n = \prod_{\substack{a+b=n\\ a\leqslant 0}} C_{a,b} \oplus \bigoplus_{\substack{a+b=n\\ a>0}} C_{a,b} = \bigoplus_{\substack{a+b=n\\ b\leqslant 0}} C_{a,b} \oplus \prod_{\substack{a+b=n\\ b>0}} C_{a,b}.$$

The differential on $\mathrm{Tot}C$ is given, for $c \in (\mathrm{Tot}C)_n$, by

$$(dc)_a = \sum_{i\geqslant 0} d_i(c)_{a+i},$$

where $(c)_a$ denotes the projection of c to $C_{a,*} = \prod_b C_{a,b}$.

Since $(c)_j = 0$ for j sufficiently large, the sum above is finite and also $(dc)_a = 0$ for sufficiently large a, so this formula determines a well-defined map on $\mathrm{Tot}C$.

Note that it is not possible in general to consider a direct product total complex with $\prod_{a+b=n} C_{a,b}$ in degree n, since in this case the formula above can involve infinite sums.

Given a multicomplex C, we consider the filtered complex D, where $D := \mathrm{Tot}C$ filtered by the subcomplexes

$$(F_pD)_n = \prod_{\substack{a+b=n\\ a\leqslant p}} C_{a,b}.$$

Note that $F_pD = \bigoplus_{i=0}^{r-1} C_{p-i,*} \oplus F_{p-r}D$. Consequently, an element $x \in F_pD$ can be written

$$x = (x)_p + (x)_{p-1} + \ldots + (x)_{p-(r-1)} + u \tag{1}$$

with $u \in F_{p-r}D$, where $(x)_{p-i}$ is the projection of x to $C_{p-i,*}$.

We consider the spectral sequence associated to this filtered complex, as presented in [De71, 1.3]. For $r \geqslant 0$, the r-stage $E_r(D)$ is an r-bigraded complex – that is, a bigraded module endowed with a square zero map δ_r of bidegree $(-r, r-1)$ – and may be written as the quotient

$$E_r^{p,*}(D) \cong \mathcal{Z}_r^{p,*}(D)/\mathcal{B}_r^{p,*}(D),$$

where the *r-cycles* are given by

$$\mathcal{Z}_r^{p,*}(D) := F_pD \cap d^{-1}(F_{p-r}D)$$

and the *r-boundaries* are given by $\mathcal{B}_0^{p,*}(D) = \mathcal{Z}_0^{p-1,*}(D)$ and

$$\mathcal{B}_r^{p,*}(D) := \mathcal{Z}_{r-1}^{p-1,*}(D) + d\mathcal{Z}_{r-1}^{p+r-1,*}(D) \text{ for } r \geqslant 1.$$

Given an element $x \in \mathcal{Z}_r^{p,*}(D)$, we will denote by $[x]_r$ its image in $E_r^{p,*}(D)$. For $[x]_r \in E_r^{p,*}(D)$, we have

$$\delta_r([x]_r) = [dx]_r. \tag{2}$$

Expanding the expressions $dx \in F_{p-r}D$ and $dc = x$ for some $c \in F_{p+r-1}D$ using the decomposition (1) above leads to the following definition.

Definition 2.6. Let $x \in C_{p,*}$ and let $r \geqslant 1$. We define subgraded modules $Z_r^{p,*}$ and $B_r^{p,*}$ of $C_{p,*}$ as follows.

$$x \in Z_r^{p,*} \iff \text{for } 1 \leqslant j \leqslant r-1, \text{ there exists } z_{p-j} \in C_{p-j,*} \text{ such that}$$

$$d_0 x = 0 \text{ and } d_n x = \sum_{i=0}^{n-1} d_i z_{p-n+i}, \text{ for all } 1 \leqslant n \leqslant r-1. \quad (\star_1)$$

$$x \in B_r^{p,*} \iff \text{for } 0 \leqslant k \leqslant r-1, \text{ there exists } c_{p+k} \in C_{p+k,*} \text{ such that}$$

$$\begin{cases} x = \sum_{k=0}^{r-1} d_k c_{p+k} & \text{and} \\ 0 = \sum_{k=l}^{r-1} d_{k-l} c_{p+k} & \text{for } 1 \leqslant l \leqslant r-1. \end{cases} \quad (\star_2)$$

Proposition 2.7. *For $r \geqslant 1$ and all p, we have $B_r^{p,*} \subseteq Z_r^{p,*}$.*

Proof. Let $x \in B_r^{p,*}$, with $c_{p+k} \in C_{p+k,*}$ for $0 \leqslant k \leqslant r-1$ satisfying equations $(\star_2)$. Define

$$z_{p-j} = -\sum_{i=0}^{r-1} d_{j+i} c_{p+i} \in C_{p-j,*},$$

for $1 \leqslant j \leqslant r-1$. Direct calculation shows that these elements satisfy $(\star_1)$ and thus $x \in Z_r^{p,*}$. □

Proposition 2.8. *The map*

$$\psi : \mathcal{Z}_r^{p,*}(D)/\mathcal{B}_r^{p,*}(D) \to Z_r^{p,*}/B_r^{p,*},$$

sending $[x]_r$ to the class $[(x)_p]$, is well defined and an isomorphism.

Proof. Define

$$\hat{\psi} : \mathcal{Z}_r^{p,*}(D) \to Z_r^{p,*}/B_r^{p,*}$$

by $\hat{\psi}(x) = [(x)_p]$. To see that $(x)_p \in Z_r^{p,*}$, note that $dx \in F_{p-r}D$ implies that $(dx)_{p-n} = 0$ for all $0 \leqslant n \leqslant r-1$. Therefore $d_0(x)_p = 0$ and

$$d_n(x)_p + \sum_{i=0}^{n-1} d_i(x)_{p-n+i} = (dx)_{p-n} = 0,$$

for all $1 \leqslant n \leqslant r-1$. So, taking $z_{p-n+i} = -x_{p-n+i}$ in Definition 2.6, we see $(x)_p \in Z_r^{p,*}$ and a similar argument proves that $\hat{\psi}$ is surjective.

Let us compute its kernel. Let $x = (x)_p + w \in \operatorname{Ker}\hat{\psi}$, with $w \in F_{p-1}D$. By assumption $(x)_p \in B_r^{p,*}$, and hence for $0 \leqslant k \leqslant r-1$ there exists $c_{p+k} \in C_{p+k,*}$ such that

$$\begin{cases} (x)_p = \sum_{k=0}^{r-1} d_k c_{p+k} & \text{and} \\ 0 = \sum_{k=l}^{r-1} d_{k-l} c_{p+k}, & \text{for } 1 \leqslant l \leqslant r-1. \end{cases}$$

Let $c = \sum_{k=0}^{r-1} c_{p+k} \in F_{p+r-1}D$. The above relations imply that $(dc)_{p+l} = 0$ for all $1 \leqslant l \leqslant r-1$, and $(dc)_p = (x)_p$. Therefore, $dc \in F_pD$ and $c \in \mathcal{Z}_{r-1}^{p+r-1,*}(D)$. In addition, $(x)_p - dc \in F_{p-1}D$, and $x = dc + \rho$, where $\rho = (x)_p - dc + w \in F_{p-1}D$. Then $d^2c = 0$ implies that $dx = d\rho \in F_{p-r}D$, and hence $\rho \in \mathcal{Z}_{r-1}^{p-1,*}(D)$. Thus $\operatorname{Ker}\hat{\psi} \subseteq \mathcal{B}_r^{p,*}(D)$.

Conversely if $x \in \mathcal{B}_r^{p,*}(D)$, then $x = \rho + dc$ for some $\rho \in \mathcal{Z}_{r-1}^{p-1,*}(D)$ and some $c \in \mathcal{Z}_{r-1}^{p+r-1,*}(D)$. So, $\rho \in F_{p-1}D$ and $dc \in F_pD$. Thus, $(x)_p = (dc)_p$ and $(dc)_s = 0$ for all $s > p$. This implies that $(x)_p \in B_r^{p,*}$ and $\mathcal{B}_r^{p,*}(D) \subseteq \operatorname{Ker}\hat{\psi}$. $\square$

Remark 2.9. In the language of *witnesses* adopted in [CELW18b], the difference between the $\mathcal{Z}_r(D)$-cycles and the Z_r-cycles is essentially the difference between specifying witnesses and just requiring the existence of them. More precisely, $\mathcal{Z}_r^{p,*}(D)/F_{p-r}(D)$ corresponds to the *witness r-cycles* for split filtered complexes.

Theorem 2.10. *Under the isomorphism ψ of Proposition 2.8, the r-th differential of the spectral sequence corresponds to the map $\Delta_r : Z_r^{p,*}/B_r^{p,*} \to Z_r^{p-r,*}/B_r^{p-r,*}$ given by*

$$\Delta_r([x]) = \left[d_r x - \sum_{i=1}^{r-1} d_i z_{p-r+i} \right],$$

where $x \in Z_r^{p,*}$, and the family $\{z_{p-j}\}_{1 \leqslant j \leqslant r-1}$ satisfies $(\star_1)$.

Proof. Since $\{z_{p-j}\}_{1 \leqslant j \leqslant r-1}$ satisfies $(\star_1)$, $[x - z_{p-1} - \ldots - z_{p-r+1}]_r \in \mathcal{Z}_r^{p,*}(D)/\mathcal{B}_r^{p,*}(D)$ and

$$\psi[x - z_{p-1} - \ldots - z_{p-r+1}]_r = [x],$$

where ψ is the isomorphism from Proposition 2.8. Hence

$$\begin{aligned} \Delta_r([x]) &= \psi\delta_r([x - z_{p-1} - \ldots - z_{p-r+1}]_r) \\ &\overset{(2)}{=} \psi[d(x - z_{p-1} - \ldots - z_{p-r+1})]_r \\ &= [(d(x - z_{p-1} - \ldots - z_{p-r+1}))_{p-r}] \\ &= [d_r x - \sum_{i=1}^{r-1} d_i z_{p-r+i}]. \end{aligned}$$

$\square$

3. Examples

We revisit the examples given by Hurtubise [Hu10] in the light of the explicit description of the differentials. Hurtubise has the same sign and bidegree conventions as ours, but works with ground ring $\mathbb{Z}$.

The first two examples relate to the bicomplex case, that is multicomplexes with $d_i = 0$ for $i \geqslant 2$. The first, [Hu10, Example 1], is a "short staircase" bicomplex, giving a minimal example of non-trivial δ_2 in the spectral sequence in the bicomplex case. This may be schematically represented as

$$\begin{tikzcd} \bullet & \bullet \arrow[l] \arrow[d] & \\ & \bullet & \bullet \arrow[l] \end{tikzcd}$$

where each bullet represents a copy of $\mathbb{Z}$ and each arrow represents the identity map, the vertical one being a d_0 and the horizontal ones being d_1s. This bicomplex is (up to minor changes of convention) the bicomplex $\mathcal{ZW}_2$ of [CELW18b], a representing object for the witness 2-cycles. The second example, [Hu10, Example 2], generalizes this to a "long staircase" bicomplex, giving a minimal example of non-trivial δ_r in the spectral sequence in the bicomplex case. It can be pictured as follows.

$$\begin{tikzcd}
\bullet & \bullet \arrow[l] \arrow[d] & & & & \\
& \bullet & \bullet \arrow[l] \arrow[d] & & & \\
& & \bullet \arrow[dr, dotted, no head] & & & \\
& & & \bullet & \bullet \arrow[l] \arrow[d] & \\
& & & & \bullet & \bullet \arrow[l]
\end{tikzcd}$$

This corresponds to the bicomplex $\mathcal{ZW}_r$ of [CELW18b], a representing object for the witness r-cycles.

In [Hu10, Example 3], the first example is modified by putting in a non-trivial d_2, as indicated, with the effect that the δ_2 of the spectral sequence is then zero.

$$\begin{tikzcd}
\bullet & \bullet \arrow[l] \arrow[d] & \\
& \bullet & \bullet \arrow[l] \arrow[ull]
\end{tikzcd}$$

Finally, [Hu10, Example 4] is indicated below.

$$\begin{tikzcd}
\bullet\langle d_1 z\rangle & \bullet\,\langle d_2 y\rangle & \bullet\langle z\rangle \arrow[l, "\begin{pmatrix}1\\0\end{pmatrix}"'] \arrow[d, "1"] & & \\
& & \bullet\langle d_0 z = d_1 x\rangle & \bullet\langle x\rangle \arrow[l, "{(1,0)}"] \arrow[ull] & \bullet\,\langle y\rangle
\end{tikzcd}$$

Here the diagonal arrow is d_2 given by $\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$. Both x and y give rise to elements of Z_2, "witnessed" by z for x and by 0 for y, and our formula for Δ_2 gives

$$\Delta_2([x]) = [-d_1 z], \qquad \Delta_2([y]) = [d_2 y].$$

It is easy to see that $d_1 z \notin B_2$, so $[-d_1 z] \neq 0$. So we see that the map induced by d_2 and the second differential in the spectral sequence are both non-zero and they are different from each other.

We also revisit the original example given by Wall [Wa61]. Let the group G be an extension of a normal subgroup K by its quotient group H. Wall shows how to construct (inductively) a free resolution of G from free resolutions of K and H, via what he calls a "twisted tensor product". This resolution has the form of $\mathrm{Tot}C$ for C a multicomplex.

The explicit example given by Wall is for G a split extension of $K = \mathbb{Z}/r$ by $H = \mathbb{Z}/s$, with generators x, y, subject to relations

$$x^r = y^s = 1, \quad y^{-1}xy = x^t, \quad \text{with } t^s \equiv 1 \bmod r.$$

Applying his construction to the standard resolutions of the cyclic groups, he describes a (first quadrant) multicomplex whose Tot gives a free resolution for G.

Tensoring this over $\mathbb{Z}G$ with $\mathbb{Z}$ one obtains the following multicomplex, with homology of its total complex the group homology of G with integer coefficients. (Note that we switch over the order of Wall's bigradings, so that conventions match the rest of this paper.)

For $a \geqslant 0, b \geqslant 0$, $C_{a,b}$ is a free abelian group on generator $c_{a,b}$ and otherwise $C_{a,b} = 0$.

Then, for all a, b, writing $T_b = \sum_{j=0}^{s-1} t^{jb}$,

$$\begin{aligned} d_0 c_{a,2b-1} &= 0, & d_1 c_{2a,2b} &= T_b c_{2a-1,2b}, \\ d_0 c_{a,2b} &= r c_{a,2b-1}, & d_1 c_{2a,2b-1} &= -T_b c_{2a-1,2b-1}, \\ d_1 c_{2a+1,2b} &= (t^b - 1) c_{2a,2b}, & d_2 c_{a,2b} &= 0, \\ d_1 c_{2a+1,2b-1} &= -(t^b - 1) c_{2a,2b-1}, & d_2 c_{a,2b-1} &= -\frac{t^{bs} - 1}{r} c_{a-2,2b}, \end{aligned}$$

and $d_r = 0$ for $r > 2$.

As Wall notes, the associated spectral sequence degenerates at the E_2 term and he computes the group homology explicitly. From our point of view, we see that, in any bidegree where $Z_2 \neq 0$, the formula for d_2 is precisely what it has to be in order for Δ_2 to be zero.

In more detail, for $x \in Z_2$, we have $\Delta_2([x]) = [d_2 x - d_1 z]$, where $d_0 x = 0$ and z is such that $d_1 x = d_0 z$. If $b > 0$ is even, then $Z_2^{a,b} = 0$ since d_0 from this bidegree is multiplication by r which is injective, so we consider the other cases.

Suppose $x = \alpha c_{2a-1,2b-1} \in Z_2^{2a-1,2b-1}$. Then $d_0 x = 0$ and there is some $z = \beta c_{2a-2,2b}$ such that $d_1 x = d_0 z$. Now,

$$d_1 x = d_0 z \quad \Longleftrightarrow \quad -(t^b - 1)\alpha c_{2a-2,2b-1} = r\beta c_{2a-2,2b-1},$$

so we see that such a z exists if and only if r divides $(t^b - 1)\alpha$ and then $\beta = -\frac{(t^b-1)\alpha}{r}$. Then

$$d_1 z = T_b \beta c_{2a-3,2b} = -\frac{t^{bs} - 1}{t^b - 1} \frac{(t^b - 1)\alpha}{r} c_{2a-3,2b} = -\frac{t^{bs} - 1}{r} \alpha c_{2a-3,2b} = d_2 x,$$

so that $\Delta_2([x]) = 0$.

Now suppose $x = \alpha c_{2a,2b-1} \in Z_2^{2a,2b-1}$. Then $d_0 x = 0$ and there is some $z = \beta c_{2a-1,2b}$ such that $d_1 x = d_0 z$. This time,

$$d_1 x = d_0 z \quad \Longleftrightarrow \quad -T_b \alpha c_{2a-1,2b-1} = r\beta c_{2a-1,2b-1},$$

so we see that such a z exists if and only if r divides $T_b\alpha$, and then $\beta = -\frac{T_b \alpha}{r}$. Then

$$d_1 z = (t^b - 1)\beta c_{2a-2,2b} = -(t^b - 1)\frac{T_b \alpha}{r} c_{2a-2,2b} = -\frac{t^{bs} - 1}{r} \alpha c_{2a-2,2b} = d_2 x,$$

so again $\Delta_2([x]) = 0$.

Finally, we consider $x \in Z_2^{a,0}$. We have $Z_2^{2a,0} = 0$: if $x = \alpha c_{2a,0} \in Z_2^{2a,0}$ there must be a $z \in C_{2a-1,1}$ such that $d_1 x = s\alpha c_{2a-1,0} = d_0 z = 0$ and so $x = 0$.

So let $x \in Z_2^{2a-1,0}$. Then $d_0 x = 0$ and picking $z = 0$, we have $d_0 z = d_1 x = 0$. Then $d_1 z = 0 = d_2 x$.

Thus we see that $\Delta_2([x]) = [0]$, for every $x \in Z_2$.

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(M. Livernet) Univ Paris Diderot, Institut de Mathématiques de Jussieu-Paris Rive Gauche, CNRS, Sorbonne Université, 8 place Aurélie Nemours, F-75013, Paris, France
E-mail address: livernet@math.univ-paris-diderot.fr

(S. Whitehouse) School of Mathematics and Statistics, University of Sheffield, S3 7RH, England
E-mail address: s.whitehouse@sheffield.ac.uk

E-mail address: stephanie.ziegenhagen@web.de